Moments and Distribution of the NPV of a Project

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Agenda

- Introduction
- Serial projects:
  - Single cash flow after a single stage
  - Single cash flow after multiple stages
  - NPV of a serial project
  - Optimal sequence of stages
- General projects
- Conclusions

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  - Cash flows are incurred during the lifetime of the project

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  - Cash flows are incurred during the lifetime of the project
- For such settings, most of the literature has focused on determining the expected NPV (eNPV) of a project
- Higher moments/distribution of project NPV are currently determined using Monte Carlo simulation
- We develop exact, closed-form expressions for the moments of project NPV & develop an accurate approximation of the NPV distribution itself

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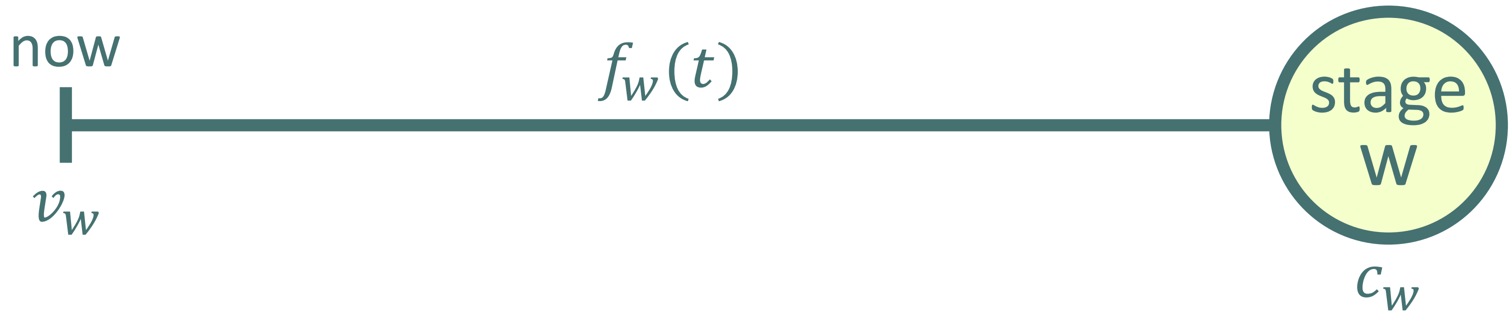

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- $f_w(t)$ = distribution of time until cash flow c_w is incurred

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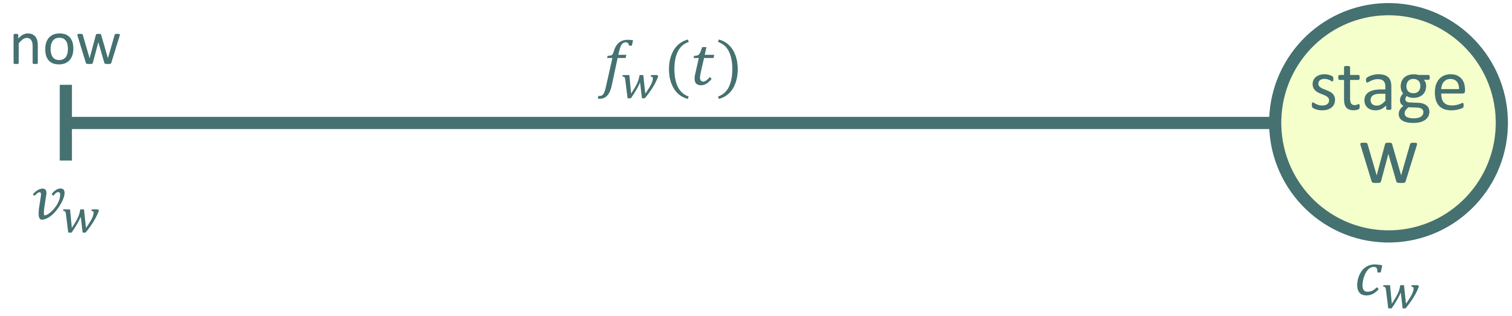

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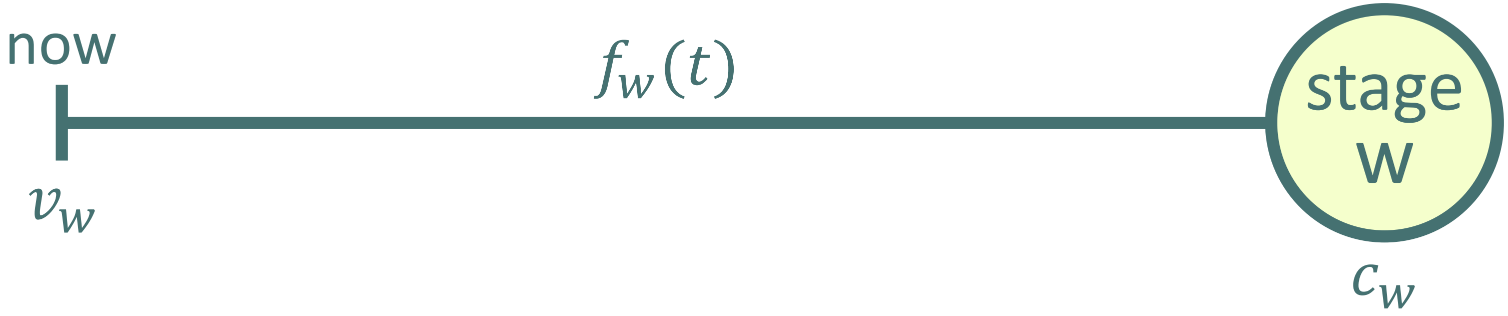

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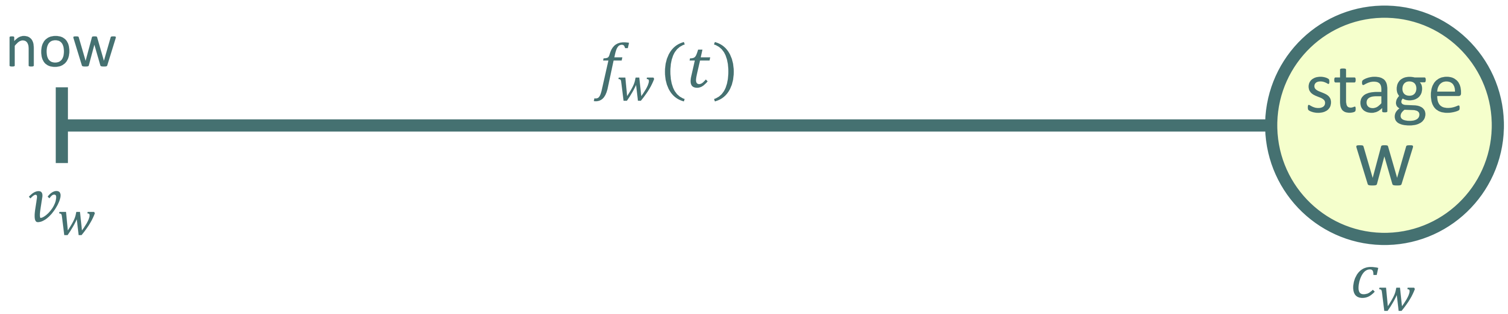

$$v_w = c_w \int_0^\infty f_w(t)\, e^{-rt}\, dt \qquad v_w = c_w M_{f_w(t)}(-r) \qquad v_w = c_w \phi_w(r)$$

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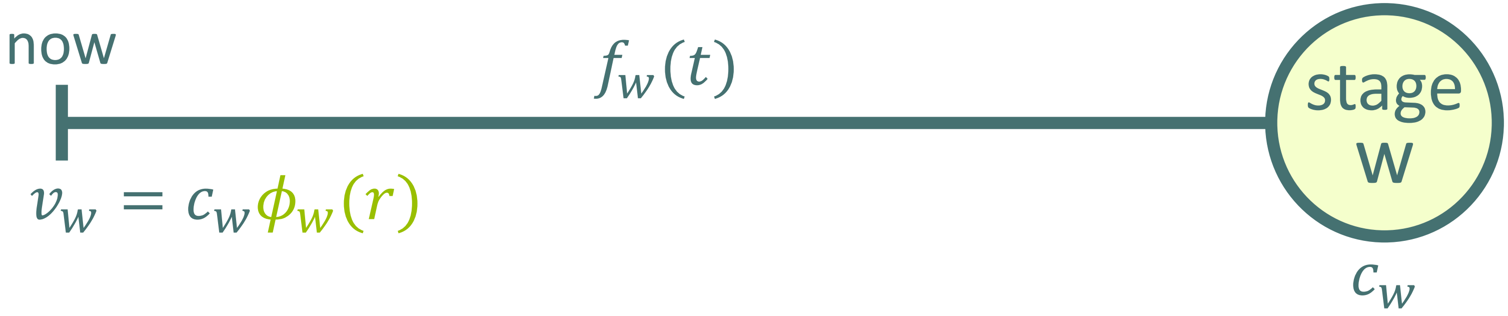

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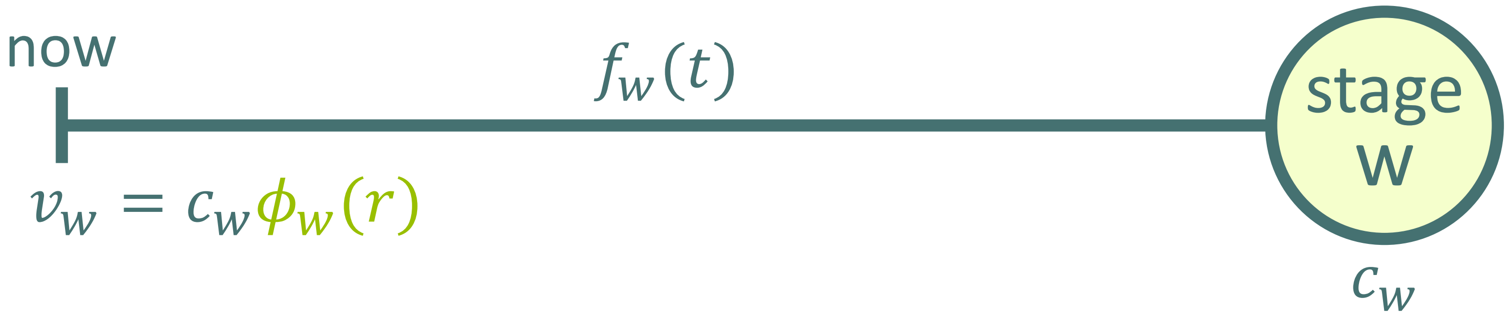

- Using discount factor $\phi_w(r)$, we can obtain the moments of the NPV:
  - $\mu_w = c_w \phi_w(r)$
  - $\sigma_w^2 = c_w^2 (\phi_w(2r) - \phi_w^2(r))$
  - $\gamma_w = c_w^3 (\phi_w(3r) - 3\phi_w(2r)\phi_w(r) + 2\phi_w^3(r))\, \sigma_w^{-3}$
  - $\theta_w = c_w^4 (\phi_w(4r) - 4\phi_w(3r)\phi_w(r) + 6\phi_w(2r)\phi_w^2(r) - 3\phi_w^4(r))\, \sigma_w^{-4}$

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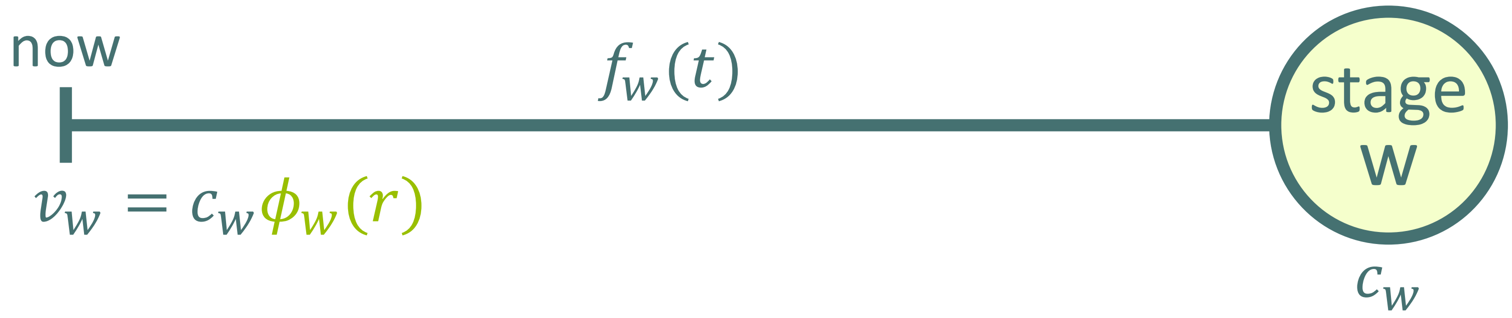

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- The CDF & PDF of the NPV of c_w are:
  - $G_w(v) = 1 - F_w\left(\ln\left(\frac{c_w}{v}\right)r^{-1}\right)$
  - $g_w(v) = \frac{f_w\left(\ln\left(\frac{c_w}{v}\right)r^{-1}\right)}{|r|v}$

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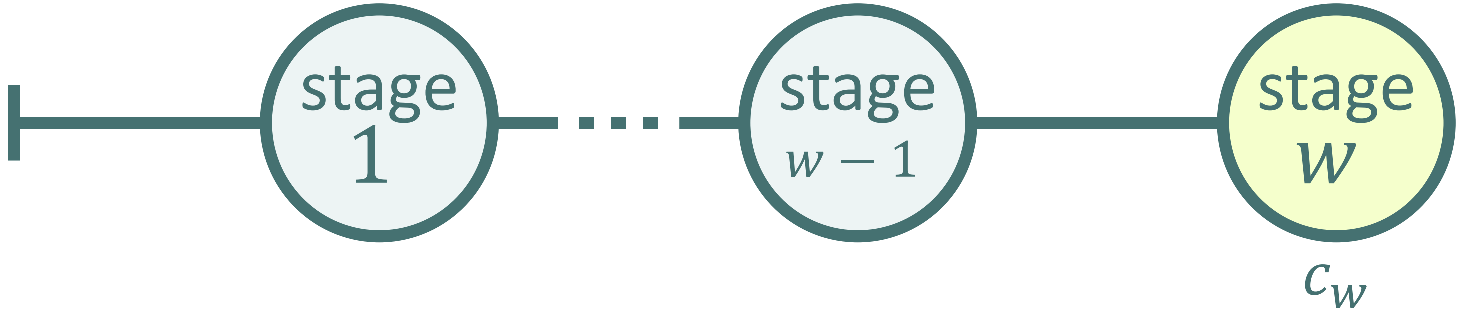

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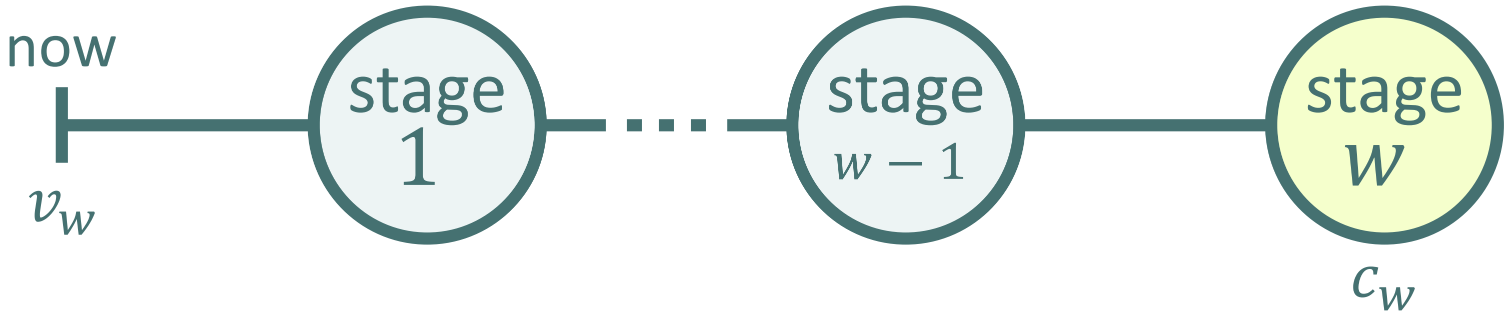

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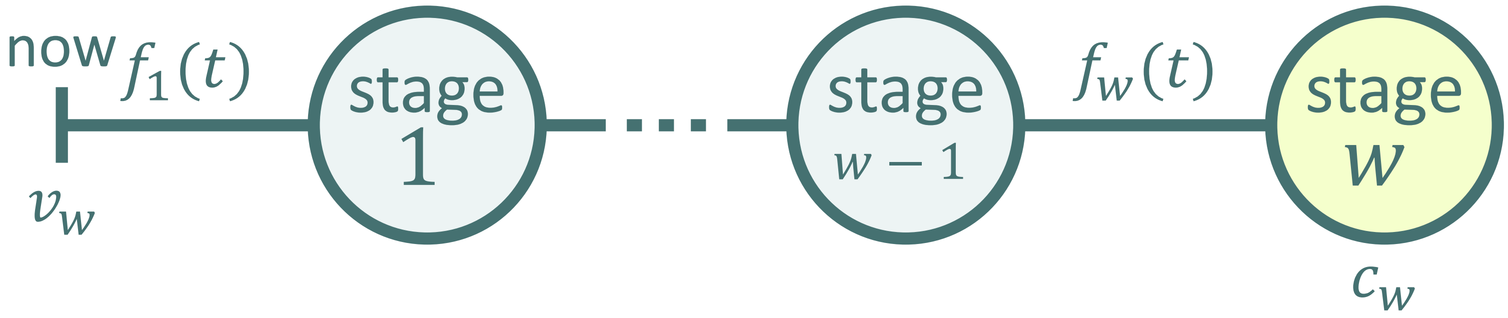

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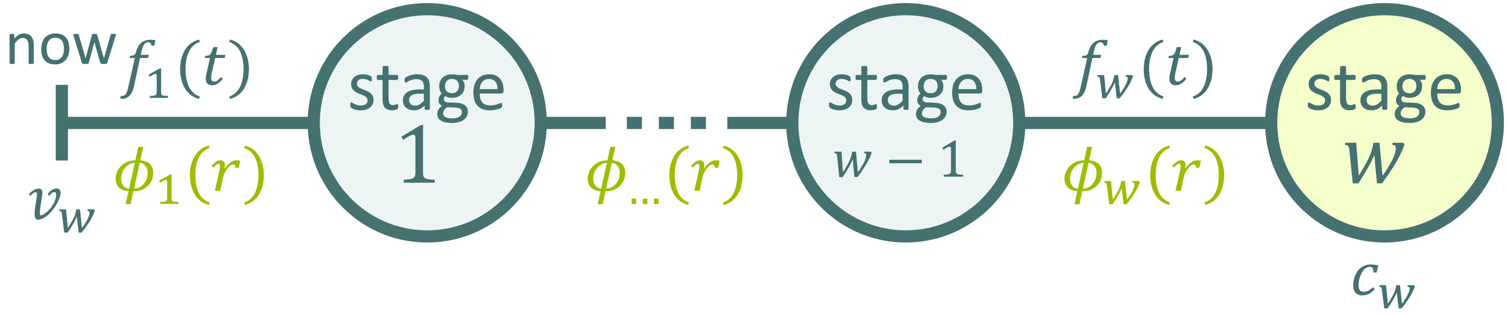

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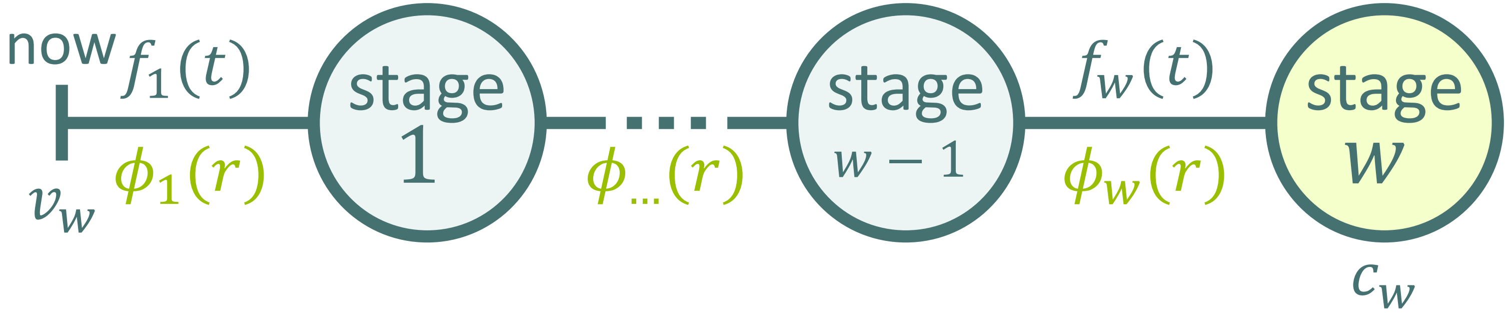

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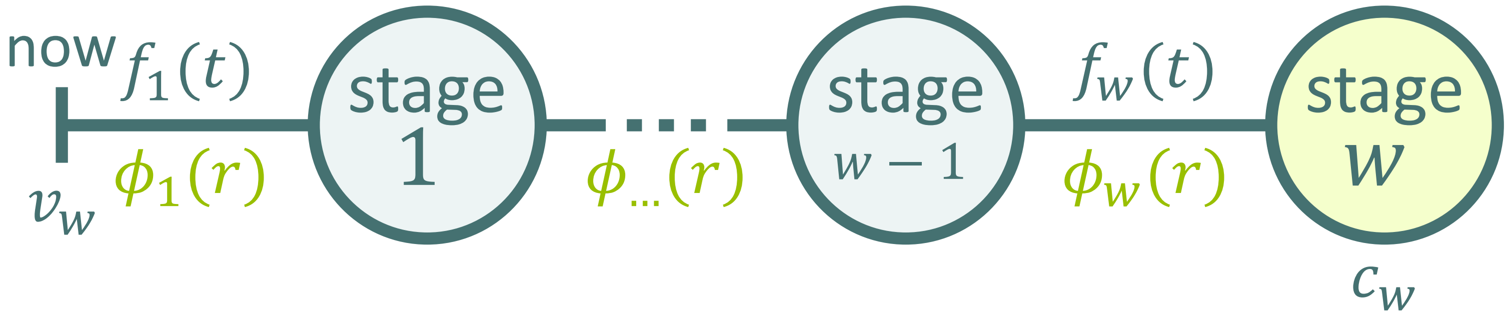

$v_w = c_w \phi_1(r) \dots \phi_w(r)$ $\quad$ $v_w = c_w \prod_{i=1}^{w} \phi_i(r)$

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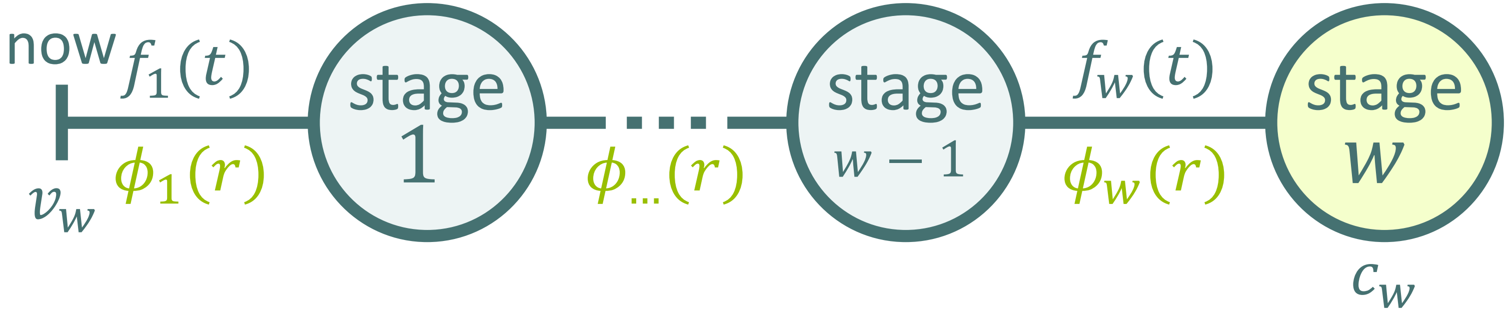

$$v_w = c_w \phi_1(r) \ldots \phi_w(r) \qquad v_w = c_w \prod_{i=1}^{w} \phi_i(r) \qquad v_w = c_w \phi_{1,w}(r)$$

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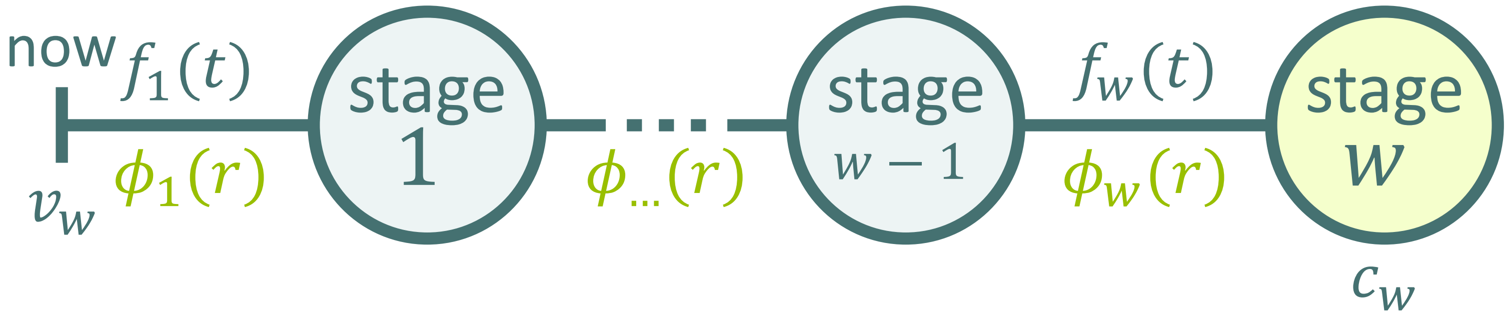

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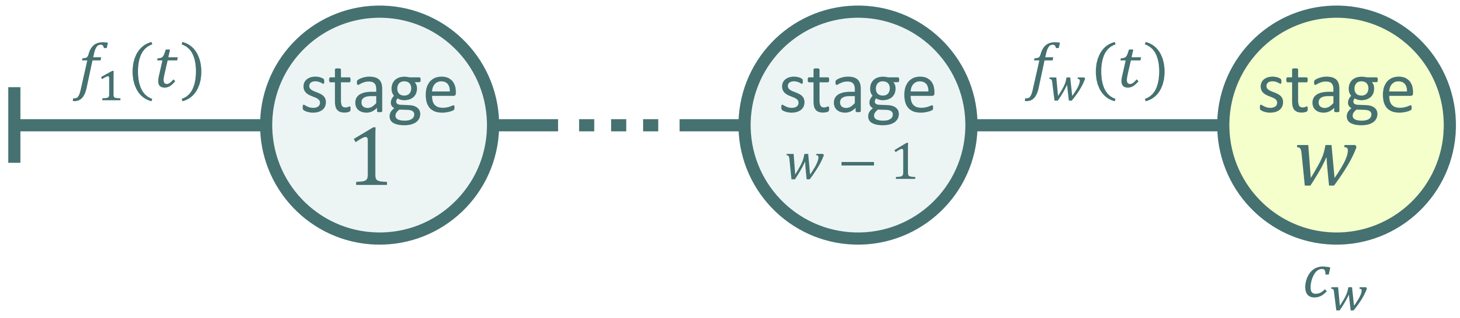

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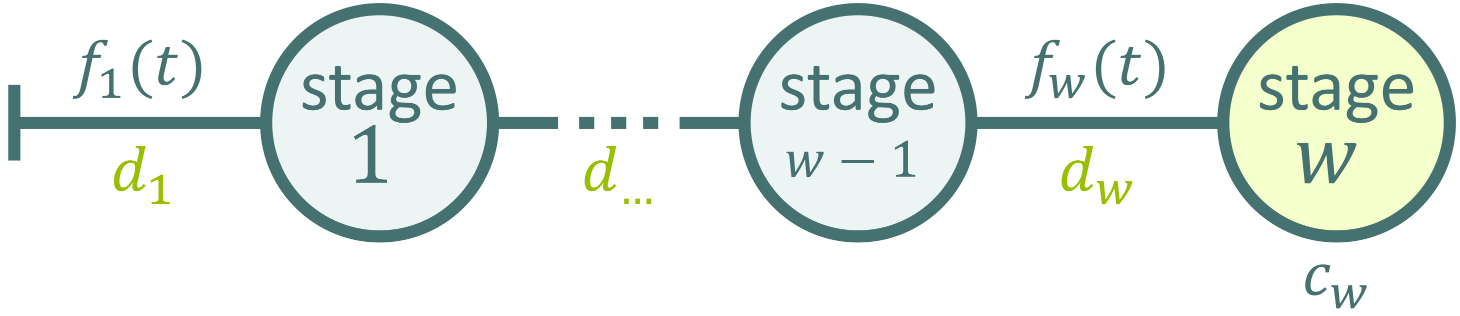

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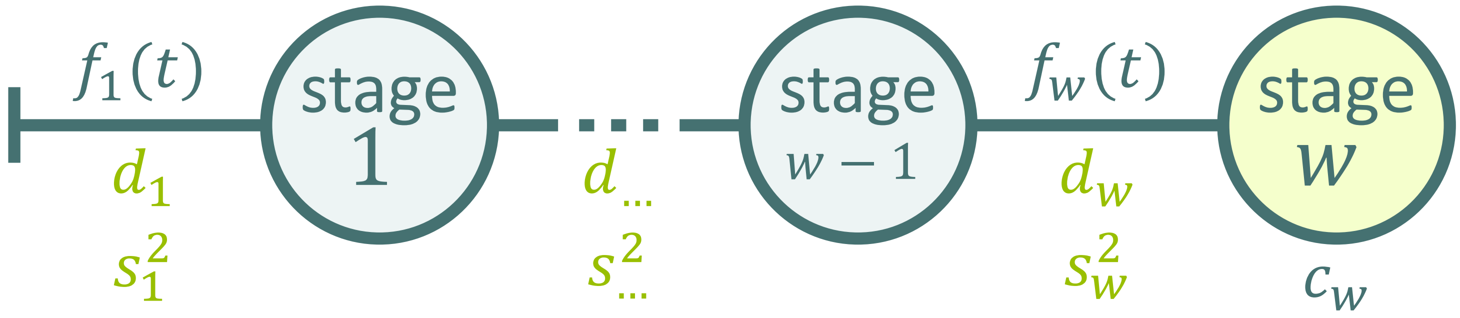

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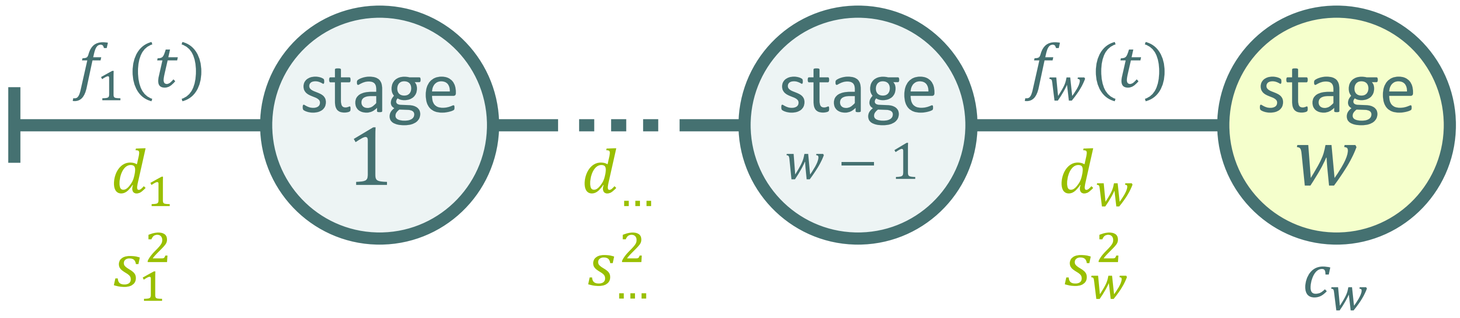

- The mean and variance of the distribution of time until cash flow c_w is incurred is:
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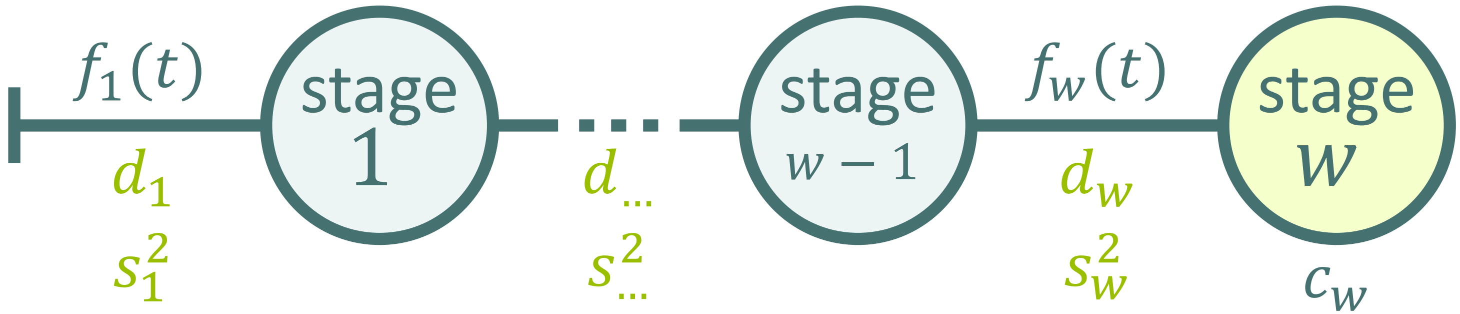

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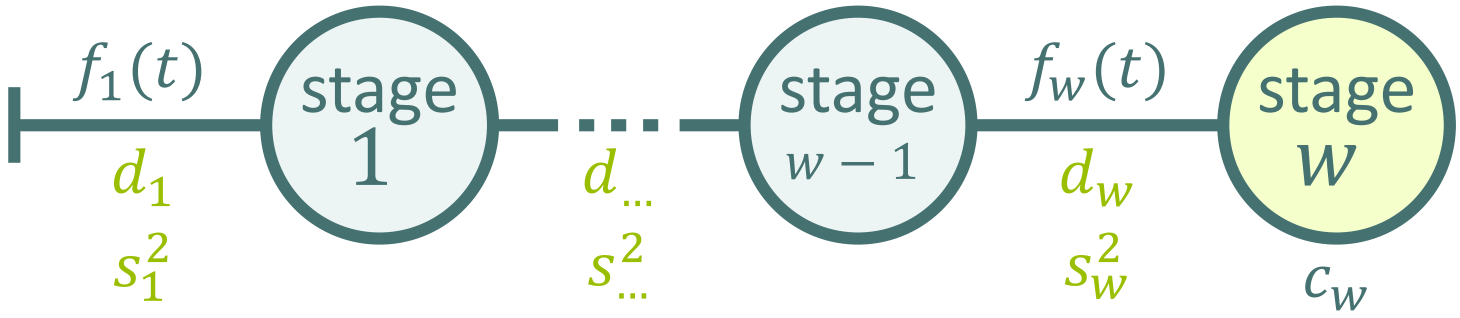

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- If $f_{1,w}(t)$ is normally distributed, the NPV of cash flow c_w is lognormally distributed!

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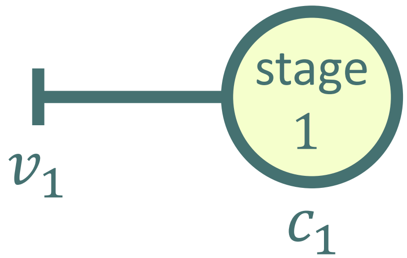

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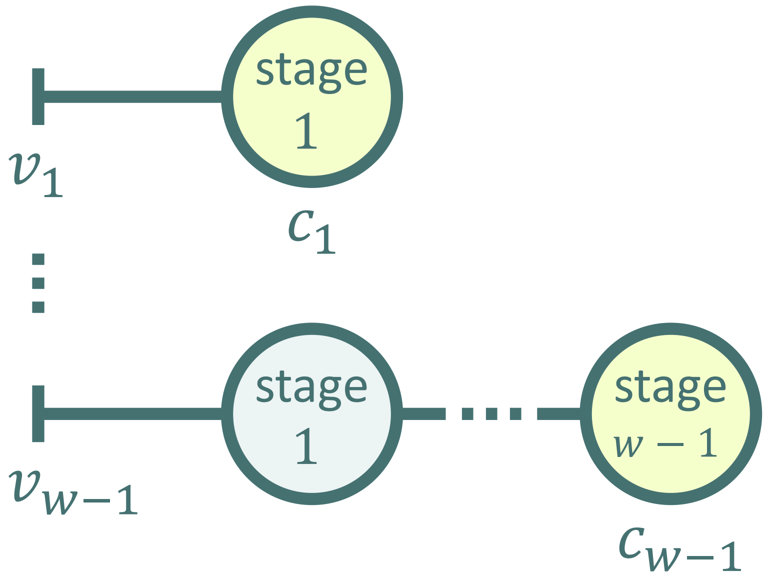

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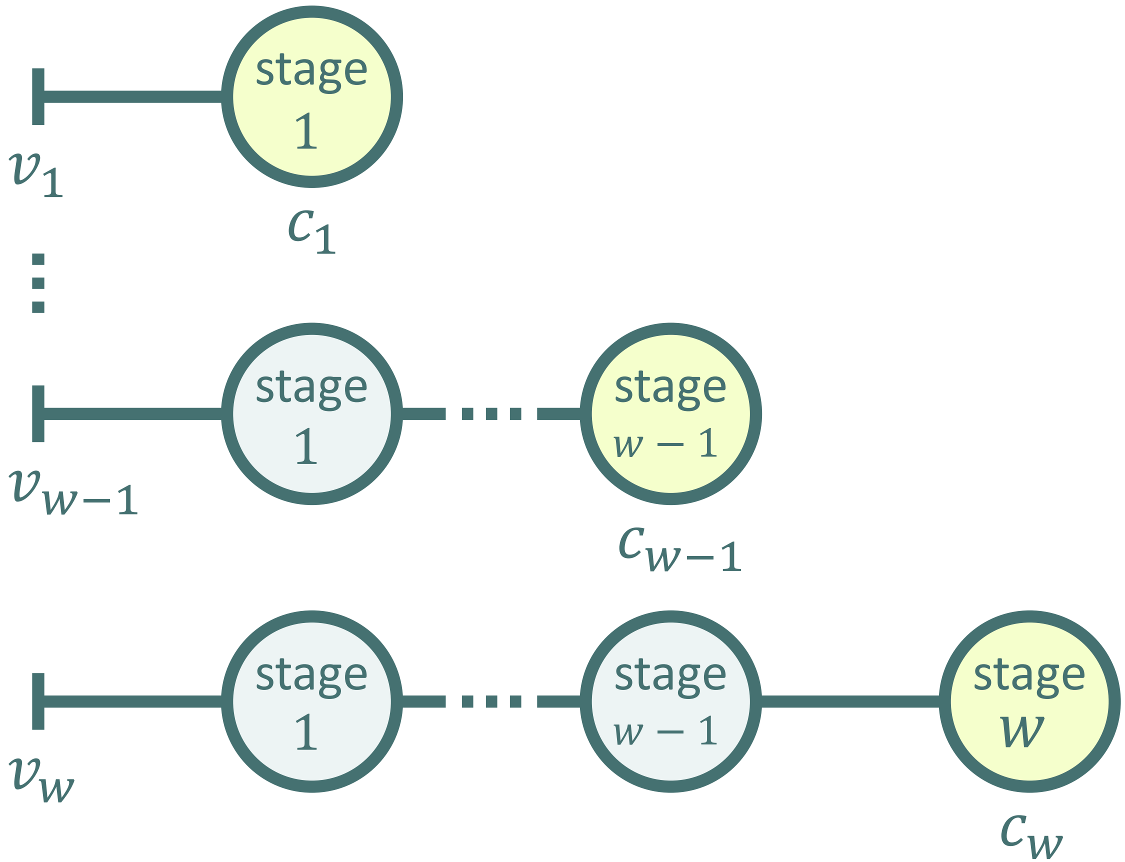

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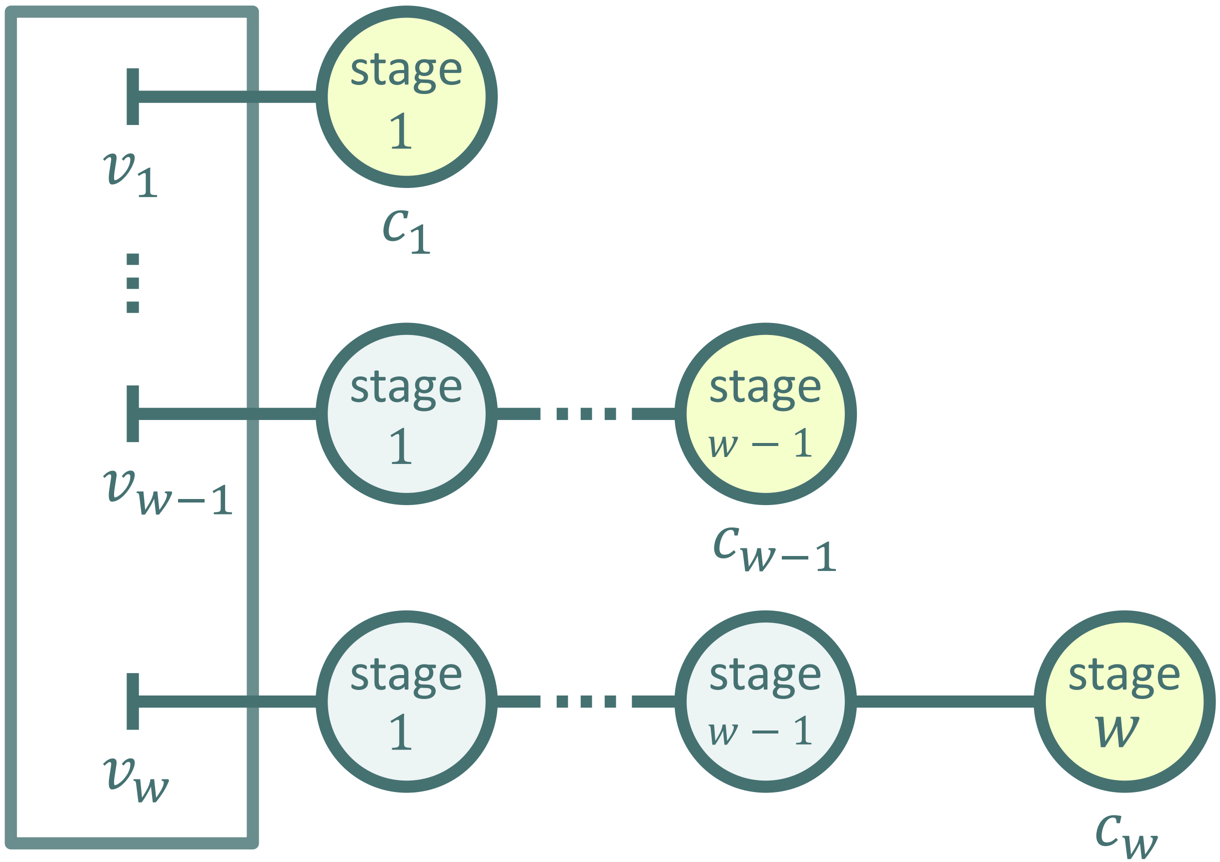

$v = v_1 + \ldots + v_{w-1} + v_w$

NPV of a serial project

We can obtain the moments of the NPV of the serial project using exact, closed-form formula’s:

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Mean μ
$\mu_w = c_w a_1$

Covariance matrix $\mathbf{\Sigma}_c$
$\mathbf{\Sigma}_c(w,w) = \sigma_w^2 = c_w^2 (a_2 - a^2)$
$\mathbf{\Sigma}_c(w,x) = c_w c_x b_1 \left(a_2 - a^2\right) = c_w^{-1} c_x b_1 \mathbf{\Sigma}_c(w,w)$

Central coskewness matrix $\mathbf{\Gamma}_c$
$\mathbf{\Gamma}_c(w,w,w) = \gamma_w \sigma_w^3 = c_w^3 \left(a_3 - 3a_2 a_1 + 2a^3\right)$
$\mathbf{\Gamma}_c(w,w,x) = c_w^{-1} c_x b_1 \mathbf{\Gamma}_c(w,w,w)$
$\mathbf{\Gamma}_c(w,x,x) = c_w c_x^2 \left(a_3 b_2 - a_2 a_1 \left(2b^2 + b_2\right) + 2a^3 b^2\right)$
$\mathbf{\Gamma}_c(w,x,y) = c_x^{-1} c_y h_1 \mathbf{\Gamma}_c(w,x,x)$

Central cokurtosis matrix $\mathbf{\Theta}_c$
$\mathbf{\Theta}_c(w,w,w,w) = \theta_w \sigma_w^4 = c_w^4 \left(a_4 - 4a_3 a_1 + 6a_2 a^2 - 3a^4\right)$
$\mathbf{\Theta}_c(w,w,w,x) = c_w^{-1} c_x b_1 \mathbf{\Theta}_c(w,w,w,w)$
$\mathbf{\Theta}_c(w,w,x,x) = c_w^2 c_x^2 \left(a_4 b_2 - 2a_3 a_1 \left(b_2 + b^2\right) + a_2 a^2 \left(b_2 + 5b^2\right) - 3a^4 b^2\right)$
$\mathbf{\Theta}_c(w,x,x,x) = c_w c_x^3 \left(a_4 b_3 - a_3 a_1 \left(b_3 + 3b_2 b_1\right) + 3a_2 a^2 \left(b_2 b_1 + b^3\right) - 3a^4 b^3\right)$
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$\mathbf{\Theta}_c(w,x,y,y) = c_w c_x c_y^2 \left(\left(a_4 - a_3 a_1\right) b_3 h_2 - \left(h_2 + 2h^2\right) \left(\left(a_3 a_1 - a_2 a^2\right) b_2 b_1\right) + \left(a_2 a^2 - a^4\right) 3b^3 h^2\right)$
$\mathbf{\Theta}_c(w,x,y,z) = c_y^{-1} c_z o_1(r) \mathbf{\Theta}_c(w,x,y,y)$

$a_i = \phi_{1,w-1}(ir)$	$b_i = \phi_{w,x-1}(ir)$	$h_i = \phi_{x,y-1}(ir)$	$o_i = \phi_{y,z-1}(ir)$
$a^i = \phi^i_{1,w-1}(r)$	$b^i = \phi^i_{w,x-1}(r)$	$h^i = \phi^i_{x,y-1}(r)$	

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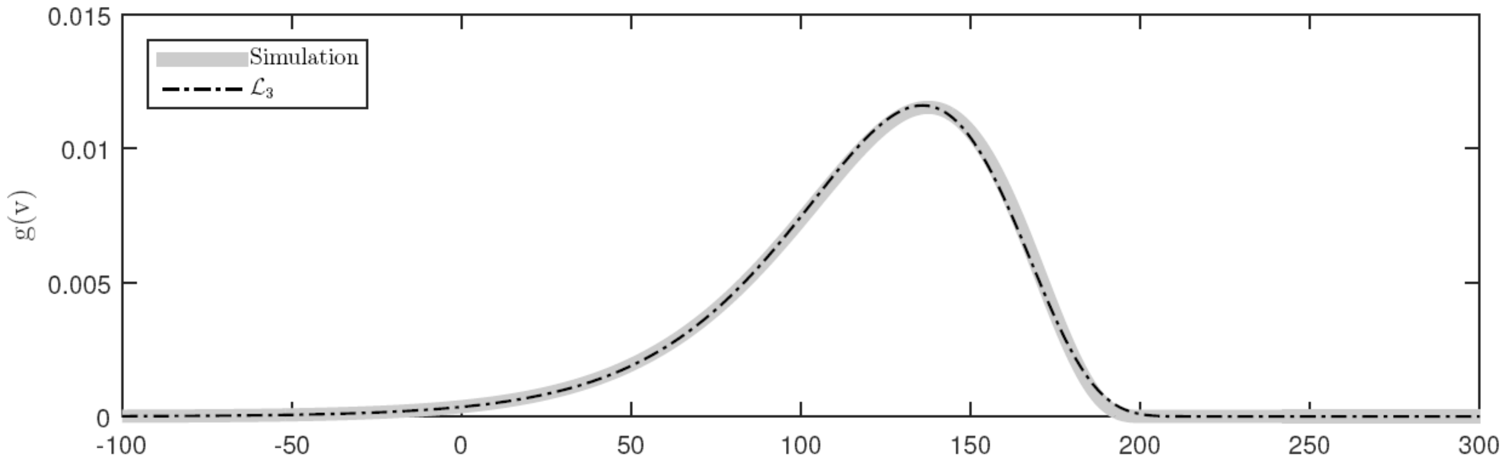
The example below illustrates the accuracy of the three-parameter lognormal distribution ($\mathcal{L}_3$):



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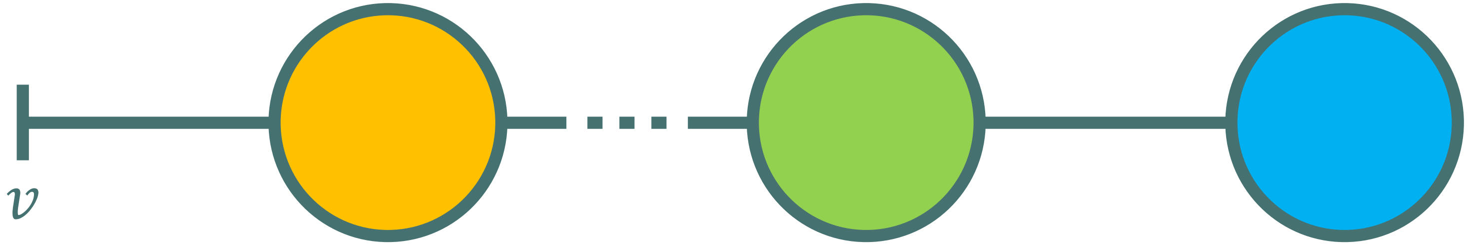

- Moments of known sequence can be obtained using exact closed-form formulas

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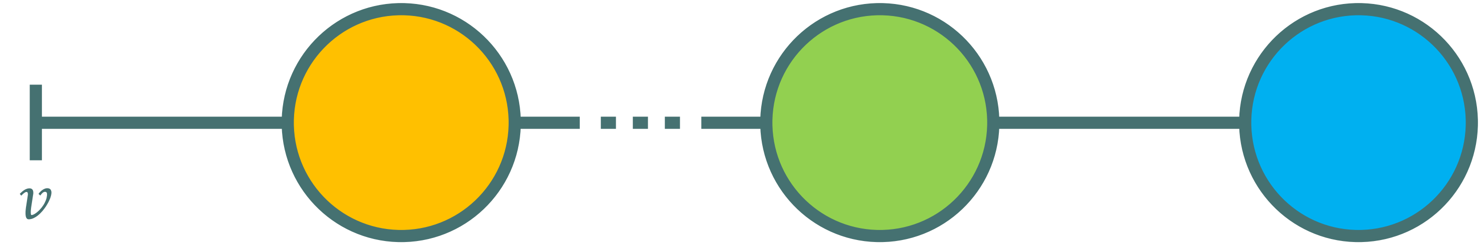
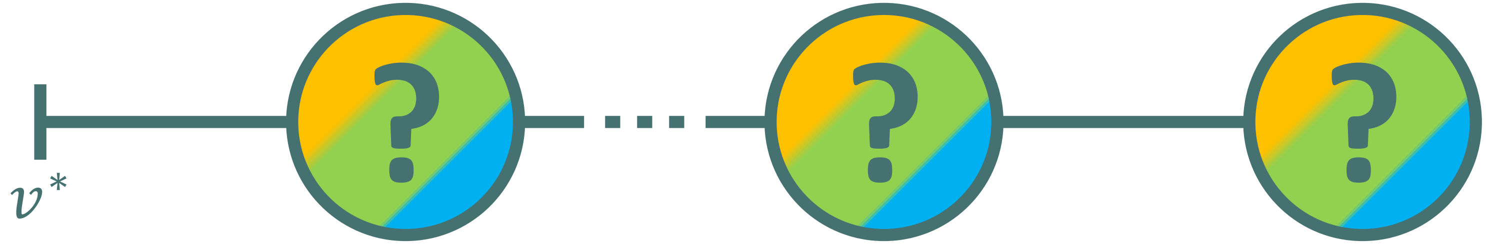


- Moments of known sequence can be obtained using exact closed-form formulas
- How to obtain the optimal sequence of a set of stages that are potentially precedence related?



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Optimal sequence of stages

- The problem to find the optimal sequence of stages is equivalent to the Least Cost Fault Detection Problem (LCFDP)
- The LCFDP minimizes the cost of the sequential diagnosis of a number of system components
- In the absence of precedence relations, the optimal sequence can be found in polynomial time
- Efficient algorithms are available for the general case

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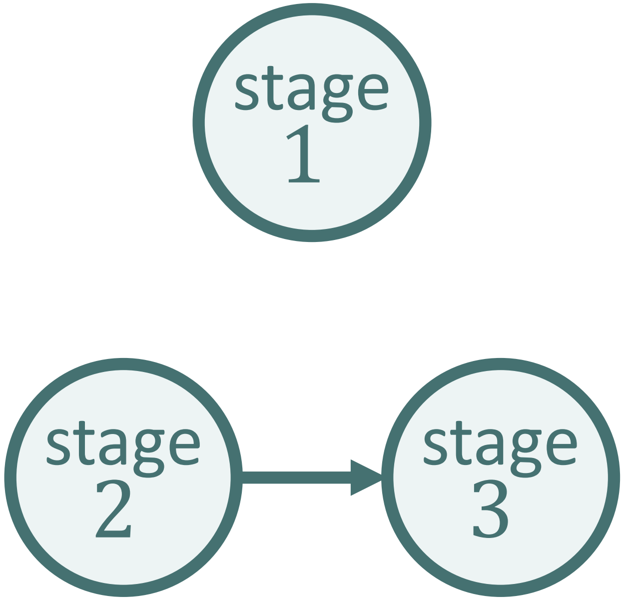
Scheduling policies



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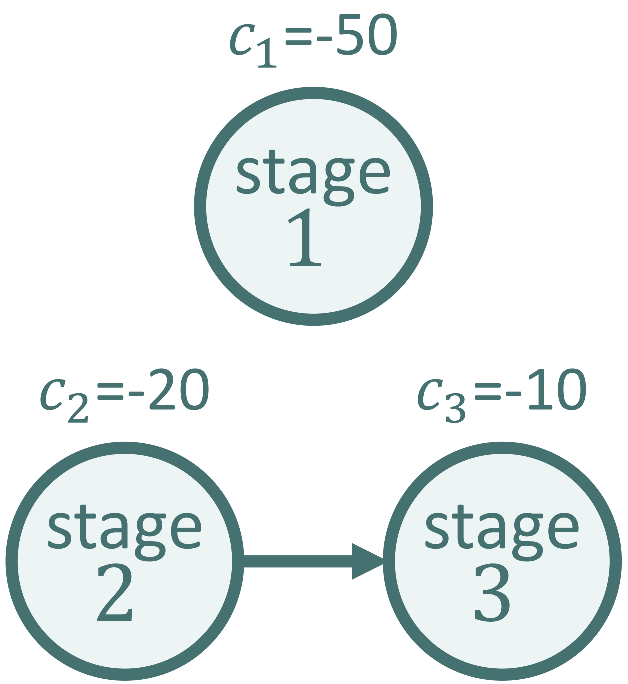

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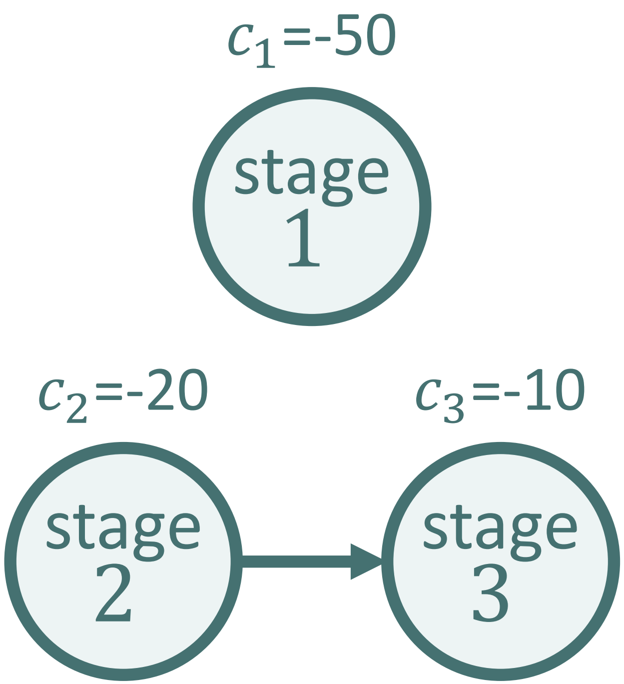

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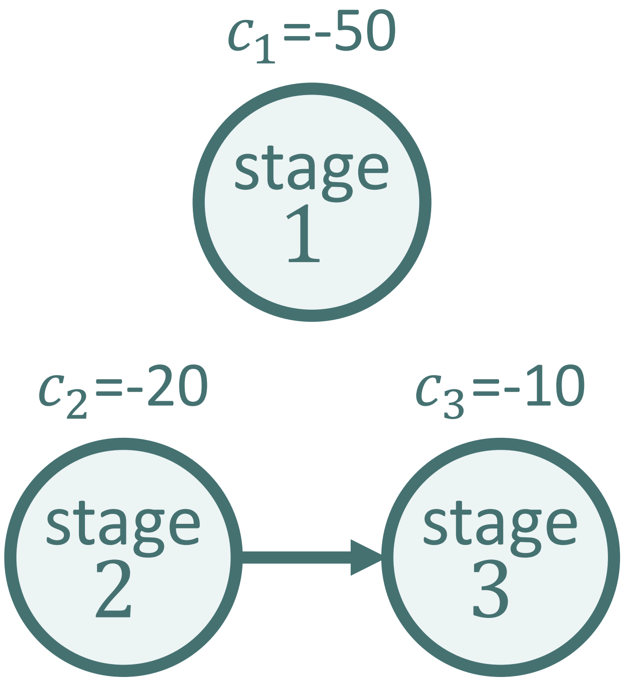

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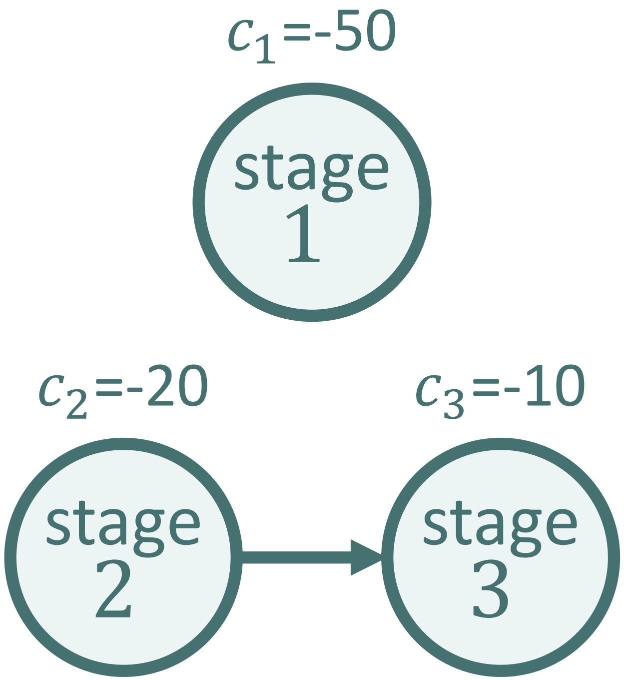

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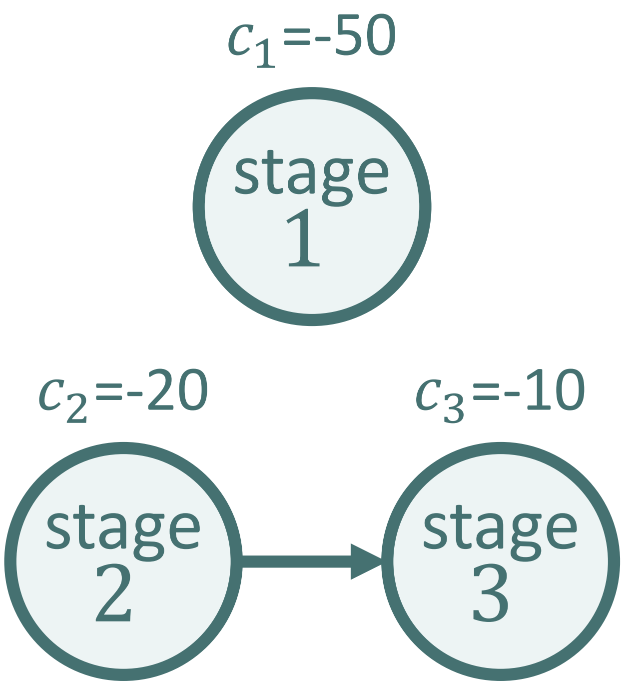
NPV of a general project
Scheduling policies



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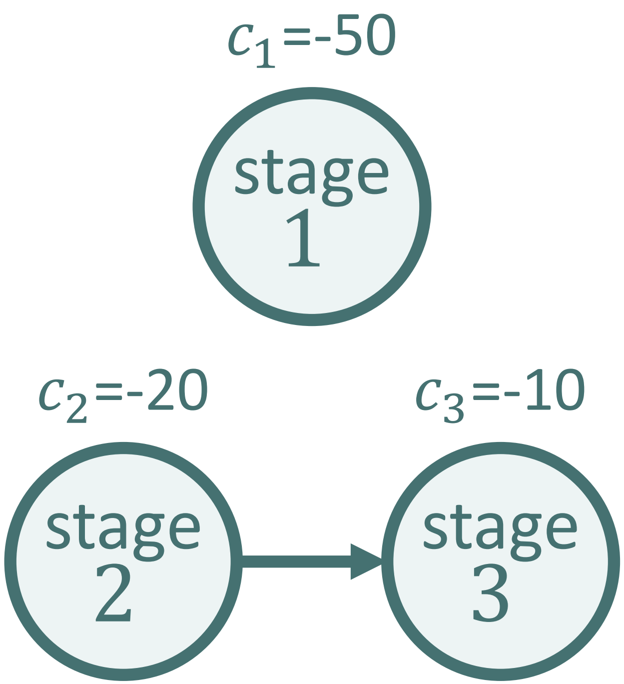

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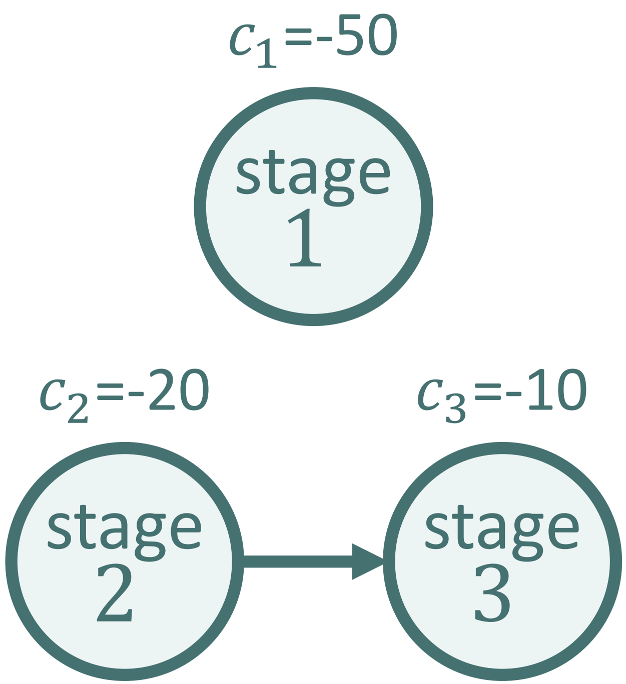

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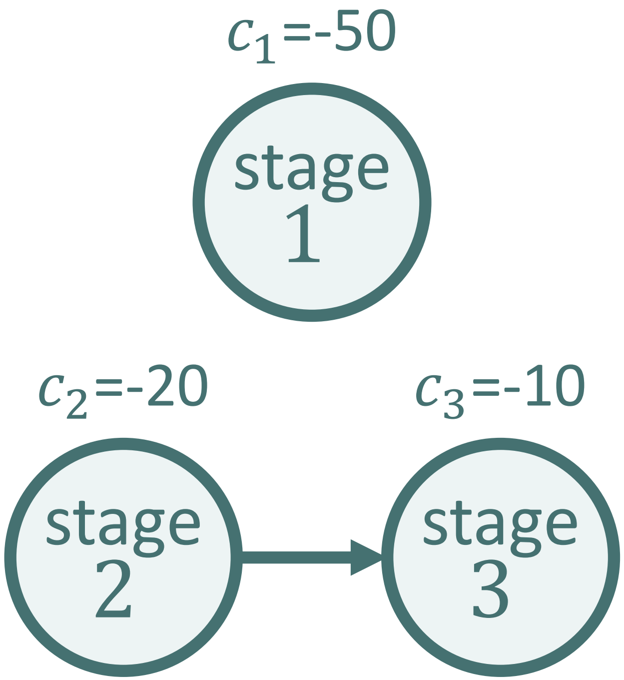
…

- Optimal policy: Start 2. Start 1 & 3 upon completion of 2.

NPV of a general project

Early-Start policy



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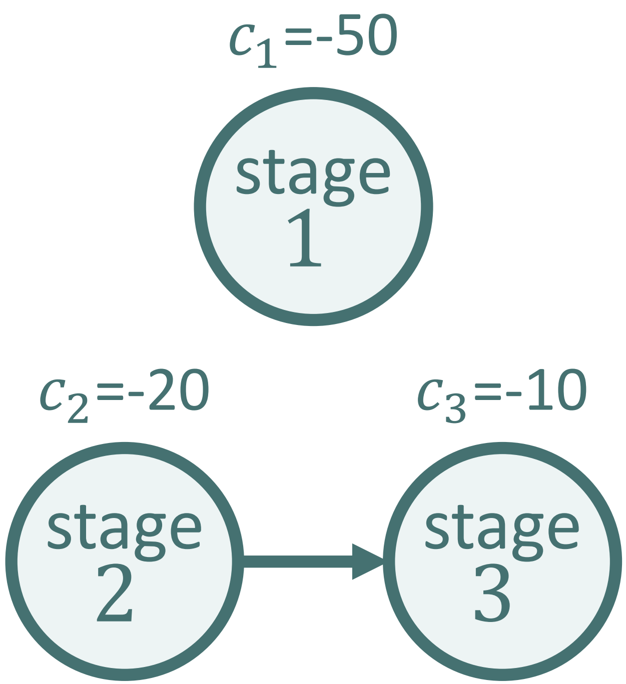

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NPV of a general project

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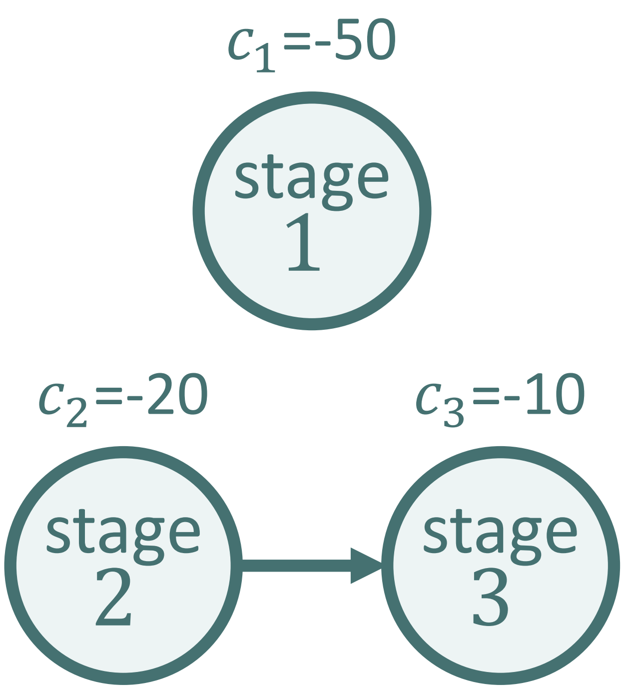

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NPV of a general project

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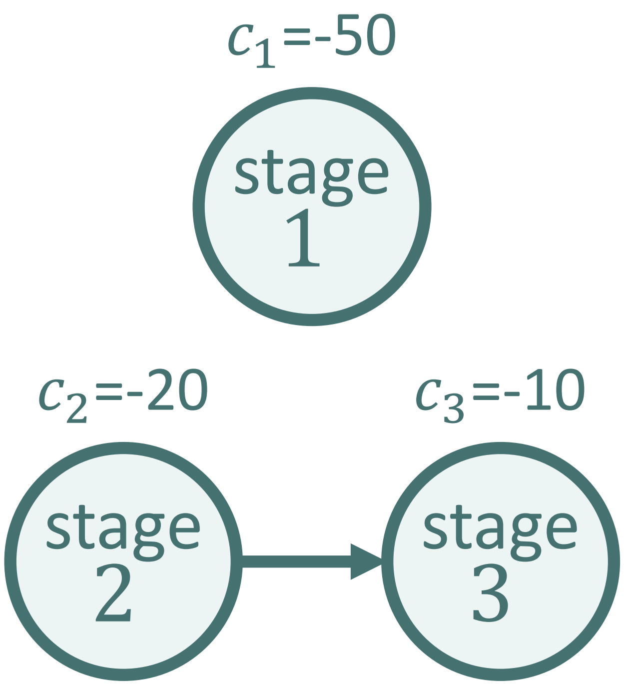

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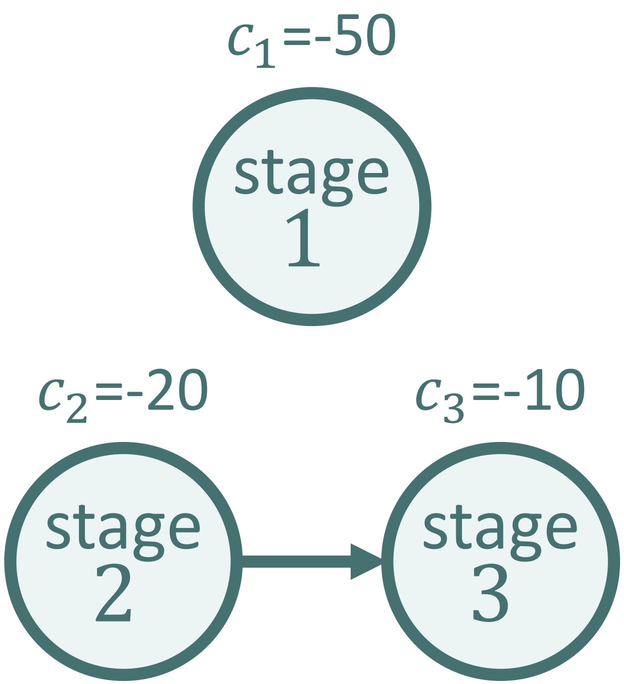

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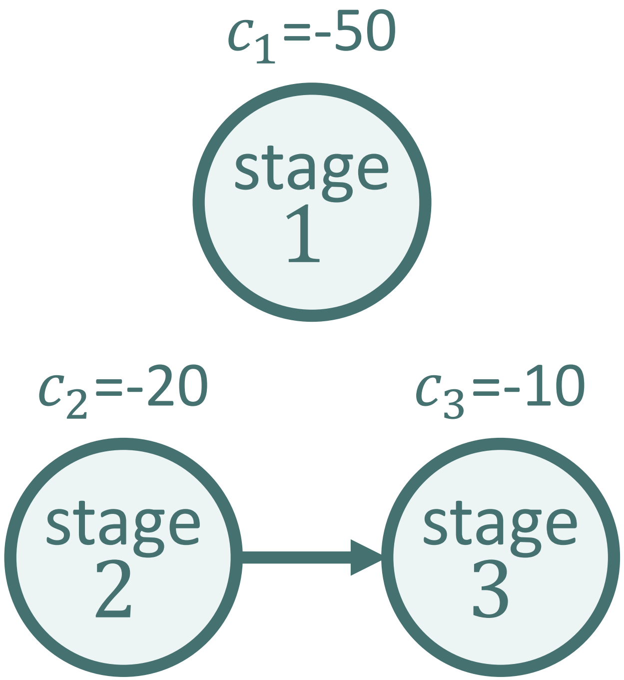
$\Rightarrow$Approximations are required!

NPV of a general project

Optimal policy



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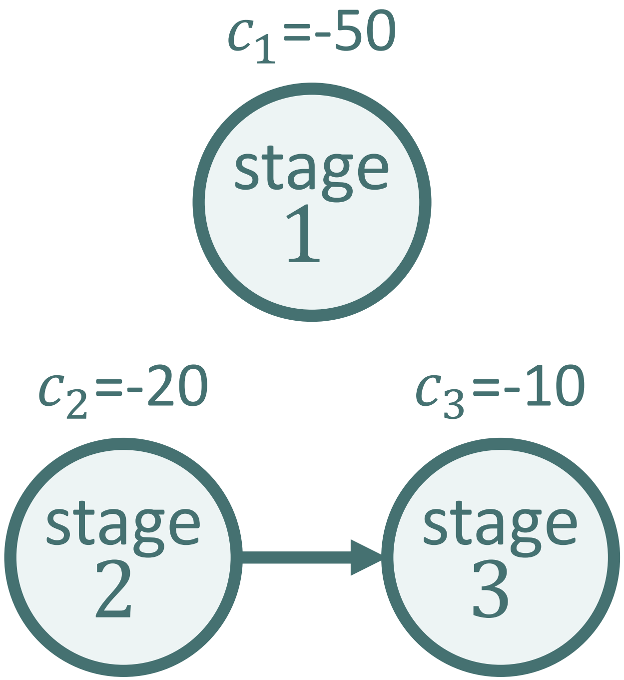

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NPV of a general project

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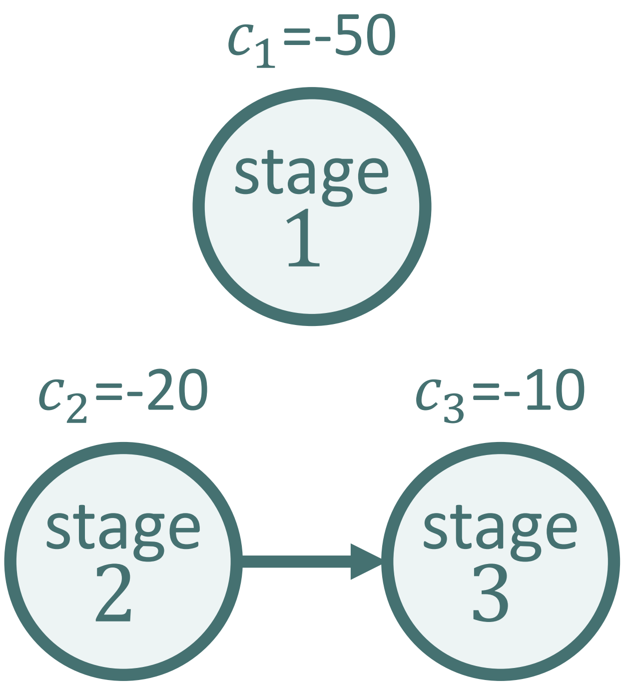

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NPV of a general project

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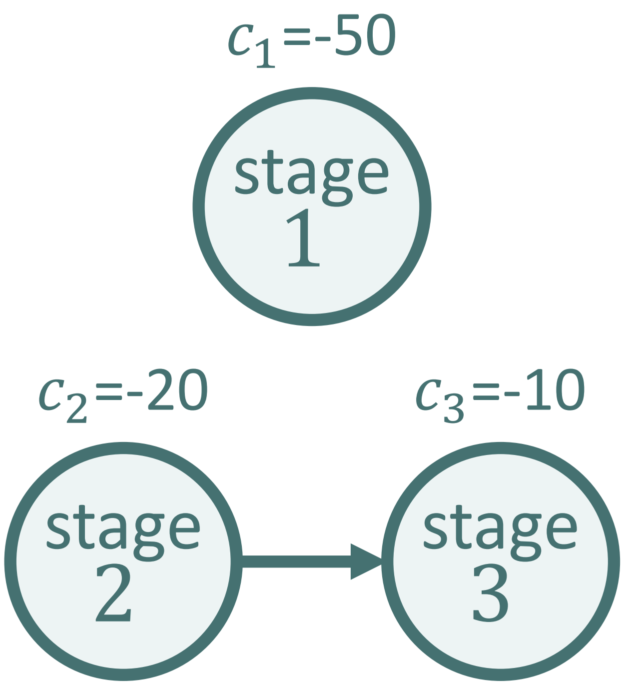

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NPV of a general project

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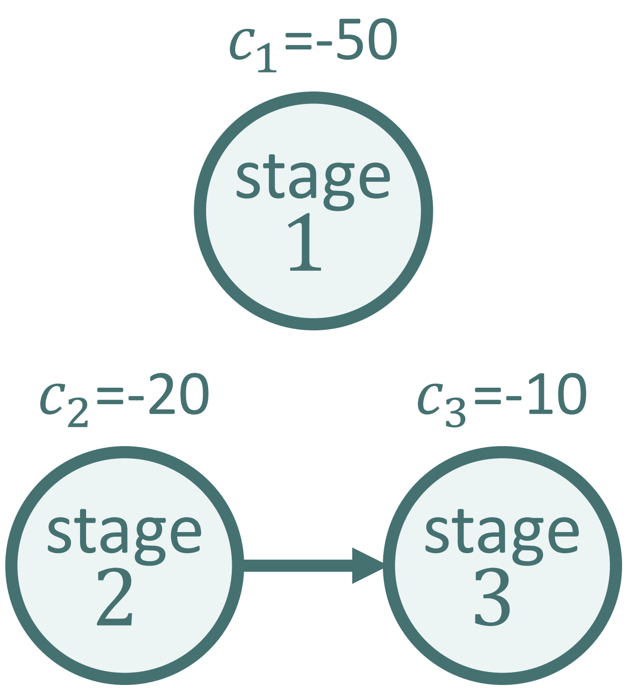

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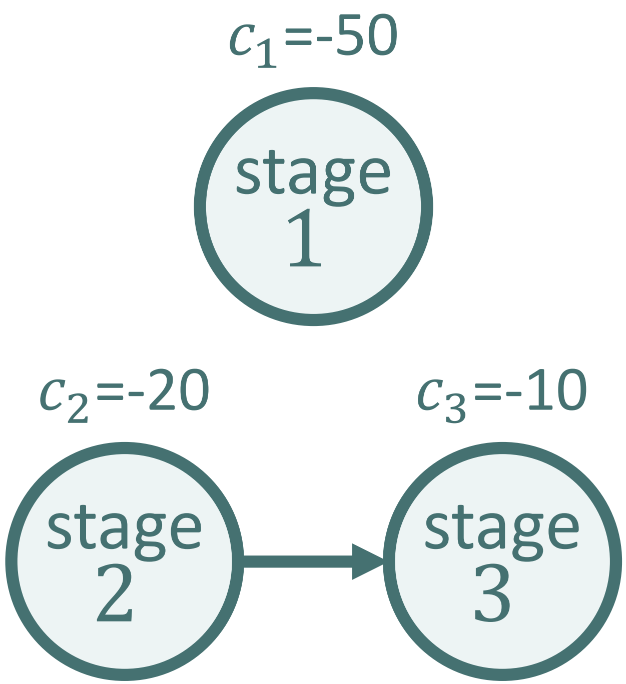
NPV of a general project

Optimal policy



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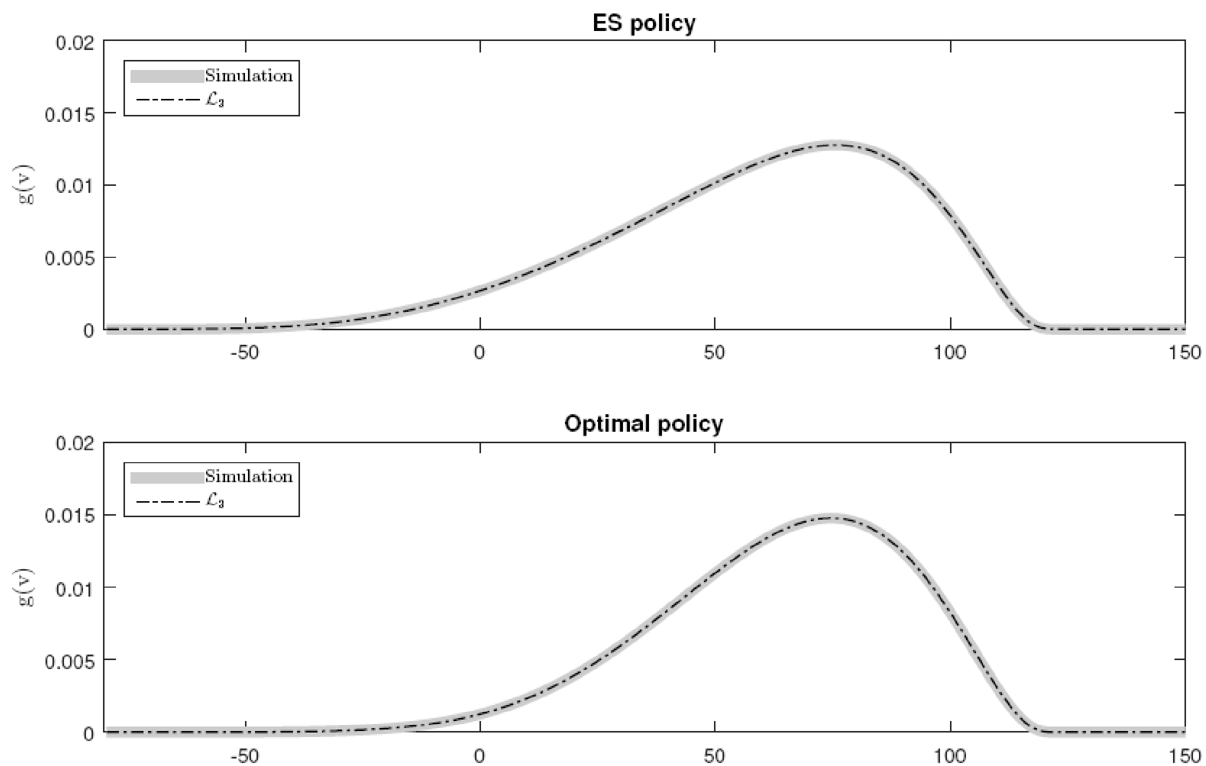
$\Rightarrow$ If this is not possible, approximations are required!

NPV of a general project

The example below illustrates the accuracy of the three-parameter lognormal distribution ($\mathcal{L}_3$) for the ES and the optimal policy:



Agenda

- Introduction
- Serial projects:
  - Single cash flow after a single stage
  - Single cash flow after multiple stages
  - NPV of a serial project
  - Optimal sequence of stages
- General projects
- Conclusions

Conclusion

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Conclusion

- We obtain exact, closed-form expressions for the moments of the NPV of serial projects
- The distribution of the NPV of a serial project can be approximated accurately using a three-parameter lognormal distribution
- The optimal sequence of stages can be found efficiently
- The eNPV of a general project can be obtained using exact, closed-form expressions
- Higher moments & the distribution of the NPV of a general project can be approximated

?